Solution to Problem 7) Since the proofs for right and left eigen-vectors are essentially the same, the argument that follows considers only the right eigen-vectors.

Consider the $N \times N$ matrix A and its two (distinct) eigen-values λ_1 and λ_2, whose corresponding (right) eigen-vectors are V_1 and V_2. Neither vector equals zero, because, by definition, eigen-vectors are nonzero. The two vectors will be linearly dependent if and only if $V_2 = cV_1$, where $c \neq 0$ is some constant. We will then have $AV_2 = cAV_1$, or $\lambda_2 V_2 = c\lambda_1 V_1$. Now, if $\lambda_2 = 0$, the preceding equation implies that $c = 0$, which contradicts our initial assumption that $c \neq 0$ (recalling that $\lambda_1 \neq 0$, since, by assumption, $\lambda_1 \neq \lambda_2$). If $\lambda_2 \neq 0$, then $V_2 = (c\lambda_1/\lambda_2)V_1$, where the proportionality constant between V_1 and V_2 is now $c\lambda_1/\lambda_2 \neq c$. It is thus clear that V_1 and V_2 must be linearly independent.

Let us now suppose that the linear combination $c_1V_1 + c_2V_2$ of the first two eigen-vectors, where c_1 and c_2 are two (generally complex-valued) constants, equals a third eigen-vector V_3 whose distinct eigen-value is denoted by λ_3. We will have

$$A(c_1V_1 + c_2V_2) = AV_3 \quad \rightarrow \quad c_1\lambda_1V_1 + c_2\lambda_2V_2 = \lambda_3(c_1V_1 + c_2V_2)$$

$$\rightarrow \quad c_1(\lambda_1 - \lambda_3)V_1 + c_2(\lambda_2 - \lambda_3)V_2 = 0.$$

Considering that, by assumption, $\lambda_1 - \lambda_3 \neq 0$ and $\lambda_2 - \lambda_3 \neq 0$, the only way for the above linear combination of V_1 and V_2 to vanish is for both c_1 and c_2 to equal zero. But this would imply that $V_3 = 0$, which is not acceptable. The inevitable conclusion is that V_3 cannot be expressed as a linear combination of V_1 and V_2, which is equivalent to saying that V_1, V_2, V_3 are linearly independent. In similar fashion, the argument is now extended to the remaining eigen-vectors $V_4, V_5, \cdots, V_N$, leading to the general conclusion that the eigen-vectors belonging to differing eigen-values of a square matrix are linearly independent of each other.